

On the Jeans theorem and the “Tolman-Oppenheimer-Volkov equation” in R^2 gravity

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Abstract

Corda, Mosquera Cuesta and Lorduy Gómez have shown that spherically symmetric stationary states can be used as a model for galaxies in the framework of the linearized R^2 gravity. Those states could represent a partial solution to the Dark Matter Problem. Here we discuss an improvement of this work. In fact, as the star density is a functional of the invariants of the associated Vlasov equation, we show that any of these invariants is in its turn a functional of the local energy and the angular momentum. As a consequence, the star density depends only on these two integrals of the Vlasov system. This result is known as the “*Jeans theorem*”. In addition, we find an analogous of the historical Tolman-Oppenheimer-Volkov equation for the system considered in this paper.

For the sake of completeness, in the final Section of the paper we consider two additional models which argue that Dark Matter could not be an essential element.

1 Introduction

The Dark Matter problem started in the 30’s of last century [1]. When one observes the Doppler shift of stars which move near the plane of our Galaxy

and calculates the velocities, one finds a large amount of matter inside the Galaxy which prevents the stars to escape out. That (supposed and unknown) matter generates a gravitational force very large, that the luminous mass in the Galaxy cannot explain. In order to achieve the very large discrepancy, the sum of all the luminous components of the Galaxy should be two or three times more massive. On the other hand, one can calculate the tangential velocity of stars in orbits around the Galactic center like a function of distance from the center. The result is that stars which are far away from the Galactic center move with the same velocity independent on their distance out from the center.

These strange issues generate a portion of the Dark Matter problem. In fact, either luminous matter is not able to correctly describe the radial profile of our Galaxy or the Newtonian Theory of gravitation cannot describe dynamics far from the Galactic center.

Other issues of the problem arise from the dynamical description of various self-gravitating astrophysical systems. Examples are stellar clusters, external galaxies, clusters and groups of galaxies. In those cases, the problem is similar, as there is more matter arising from dynamical analyses with respect to the total luminous matter.

Zwicky [2] found that in the Coma cluster the luminous mass is too little to generate the gravitational force which is needed to hold the cluster together [2].

The more diffuse way to attempt to solve the problem is to assume that Newtonian gravity holds at all scales that should exist and that they should exist non-luminous components which contribute to the missing mass. There are a lot of names which are used to define such non-luminous components. The Massive Compact Halo Objects (MACHOs) are supposed to be bodies composed of normal baryon matter, which do not emit (or emit little) radiation and drift through interstellar space unassociated with planetary systems [3]. They could be black holes and/or neutron stars populating the outer reaches of galaxies. The Weakly Interacting Massive Particles (WIMPs) are hypothetical particles which do not interact with standard matter (baryons, protons and neutrons) [4]. Hence, they should be particles outside the Standard Model of Particle Physics but they have not yet been directly detected. Dark Matter is usually divided in three different flavors, Hot Dark Matter (HDM) [5], Warm Dark Matter (WDM) [6] and Cold Dark Matter (CDM) [7]. HDM should be composed by ultrarelativistic particles like neutrinos. CDM should consist in MACHOs, WIMPs and axions, which are very light particles with a particular behavior of self-interaction [8]. If we consider the standard model of cosmology, the most recent results from the Planck mission [9] show that the total mass-energy of the known universe contains 4.9% ordinary matter, 26.8% Dark Matter and 68.3% Dark Energy.

An alternative approach is to explain large scale structure without dark components in the framework of Extended Theories of Gravity [10-17] and references within. In other words, we call “Dark Matter” a gravitational effect that we do not yet understand as modification to both Newtonian and Einsteinian gravity could be needed, see from [10-17]. The underlying idea in Extended Theories of Gravity is that General Relativity is a particular case of a more general effec-

tive theory which comes from basic principles [10-17]. The standard Einstein-Hilbert action of General Relativity [18, 19] is modified by adding new degrees of freedom, like high order curvature corrections (the so-called $f(R)$ theories of gravity [10-17,20] and scalar fields (the generalization of the Nordström-Jordan-Fierz-Brans-Dicke theory of gravitation [21-25], which is known as scalar-tensor gravity [26-28]). In this different context, one assumes that gravity is not scale-invariant and takes into account only the “observed” ingredients, i.e. curvature and baryon matter. Thus, it is not required to search candidates for Dark Matter which have not yet been found [10-17,20]. In this perspective, the gravitational wave astronomy should be the ultimate test for the physical consistency of General Relativity or of any other theory of gravitation [10].

2 R^2 theory of gravity and Vlasov system

In the framework of $f(R)$ theories of gravity, the R^2 theory is the simplest among the class of viable models with R^m terms. Those models support the acceleration of the universe in terms of cosmological constant or quintessence as well as an early time inflation [11, 15, 20]. Moreover, they should pass the Solar System tests, as they have an acceptable Newtonian limit, no instabilities and no Brans-Dicke problem (decoupling of scalar) in scalar-tensor version. We recall that the R^2 theory was historically proposed in [29] with the aim of obtaining the cosmological inflation.

The R^2 theory arises from the action [30]

$$S = \int d^4x \sqrt{-g} (R + bR^2 + \mathcal{L}_m), \quad (1)$$

where b represents the coupling constant of the R^2 term. In general, when the constant coupling of the R^2 term in the gravitational action (1) is much minor than the linear term R , the variation from standard General Relativity is very weak and the theory can pass the Solar System tests [30]. In fact, as the effective scalar field arising from curvature is highly energetic, the constant coupling of the R^2 non-linear term $\rightarrow 0$ [30]. In that case, the Ricci scalar, which represents an extra dynamical quantity in the metric formalism, should have a range longer than the size of the Solar System. This is correct when the effective length of the scalar field l is much shorter than the value of 0.2 mm [31]. Hence, this effective scalar field results hidden from Solar System and terrestrial experiments. By analysing the deflection of light by the Sun in the R^2 theory through a calculation of the Feynman amplitudes for photon scattering, one sees that, to linearized order, the result is the same as in standard General Relativity [22]. By assuming that the dynamics of the matter (the stars making of the galaxy) can be described by the Vlasov system, a model of stationary, spherically symmetric galaxy can be obtained. This issue was described in detail in [30], in the framework of the R^2 theory. For the sake of completeness, in this Section we shortly review this issue.

In this paper we consider Greek indices run from 0 to 3. By varying the action of Eq. (1) with respect to $g_{\mu\nu}$, one gets the field equations [30]

$$G_{\mu\nu} + b\{2R[R_{\mu\nu} - \frac{1}{4}g_{\mu\nu}R] +$$

$$-2R_{;\mu;\nu} + 2g_{\mu\nu}\square R\} = T_{\mu\nu}^{(m)}. \quad (2)$$

By taking the trace of this equation, the associated Klein - Gordon equation for the Ricci curvature scalar

$$\square R = E^2(R + T), \quad (3)$$

is obtained [30], where \square is the d'Alembertian operator and the energy term, E , has been introduced for dimensional motivations [30]

$$E^2 \equiv \frac{1}{6b}. \quad (4)$$

Hence, b is positive [30]. $T_{\mu\nu}^{(m)}$ in Eq. (2) is the standard stress-energy tensor of the matter and General Relativity is easily re-obtained for $b = 0$ in Eq. (2).

As we study interactions between stars at galactic scales, we consider the linearized theory in vacuum ($T_{\mu\nu}^{(m)} = 0$), which gives a better approximation than Newtonian theory [30]. Calling \tilde{R} the linearized quantity which corresponds to R , considering the plane wave [30]

$$b\tilde{R} = a(\vec{p}) \exp(iq^\beta x_\beta) + c.c. \quad (5)$$

with [30]

$$p^\beta \equiv (\omega, \vec{p}), \quad \omega = p \equiv |\vec{p}|$$

$$q^\beta \equiv (\omega_E, \vec{p}), \quad \omega_E = \sqrt{E^2 + p^2}. \quad (6)$$

one can choose a gauge for a gravitational wave propagating in the $+z$ direction in which a first order solution of eqs. (2) with $T_{\mu\nu}^{(m)} = 0$ is given by the the conformally flat line element [30]

$$ds^2 = [1 + b\tilde{R}(t, z)](dx^2 + dy^2 + dz^2 - dt^2), \quad (7)$$

We recall that the dispersion law for the modes of $b\tilde{R}$, i.e. the second of Eq. (6), is that of a wave-packet [30]. The group-velocity of a wave-packet of $b\tilde{R}$ centred in \vec{p} is [30]

$$\vec{v}_G = \frac{\vec{p}}{\omega_E}. \quad (8)$$

From the second of Eq. (6) and Eq. (8) one gets [30]

$$v_G = \frac{\sqrt{\omega_E^2 - E^2}}{\omega_E}, \quad (9)$$

which can be rewritten as [30]

$$E = \sqrt{(1 - v_G^2)} \omega_E. \quad (10)$$

If one assumes that the dynamics of the stars making of the galaxy is described by the Vlasov system, the gravitational forces between the stars will be mediated by the metric (7). Thus, the key assumption is that, in a cosmological framework, the wave-packet of $b\tilde{R}$ centred in \vec{p} , which is given by the (linearized) spacetime curvature, governs the motion of the stars [30]. In this way the “curvature” energy E is identified as the Dark Matter content of a galaxy of typical mass-energy $E \simeq 10^{45} g$, in ordinary c.g.s. units [30]. As $E \simeq 10^{45} g$, from Eq. (4) one gets $b \simeq 10^{-34} cm^4$ in natural units [30]. Hence, the constant coupling of the R^2 term in the action (1) is much minor than the linear term R and the variation from standard General Relativity is very weak. This implies that the theory can pass the Solar System tests as the effective length of the scalar field is $l \ll 0.2 mm$ [30].

We can use a conformal transformation [30, 32] to rescale the line-element (7) like

$$\tilde{g}_{\alpha\beta} = e^\Phi g_{\alpha\beta}, \quad (11)$$

where we set [30]

$$\Phi \equiv b\tilde{R}. \quad (12)$$

Thus, in the linearized theory we get

$$e^\Phi = 1 + b\tilde{R}. \quad (13)$$

Hence, it is the Ricci scalar, i.e. the *scalaron* [29, 30], the scalar field which translates the analysis into the conformal frame, the Einstein frame [15, 30].

Particles in a spacetime make up an ensemble with no collisions and are governed by a line element like Eq. (7) if the particle density satisfies the Vlasov equation [30,32-34]

$$\partial_t f + \frac{p^a}{p^0} \partial_{x^a} f - \Gamma_{\mu\nu}^a \frac{p^\mu p^\nu}{p^0} \partial_{p^a} f = 0, \quad (14)$$

where $\Gamma_{\mu\nu}^\alpha$ are the Christoffel coefficients, f is the particle density, p^0 is given by p^a ($a = 1, 2, 3$) according to the relation [30,32-34]

$$g_{\mu\nu} p^\mu p^\nu = -1 \quad (15)$$

Eq. (15) means that the four momentum p^μ lies on the mass-shell of the spacetime [30,32-34].

In general, the Vlasov-Poisson system is introduced through the system of equations [30,32-34]

$$\partial_t f + v \cdot \nabla_x f - \nabla_x U \cdot \nabla_v f = 0$$

$$\nabla \cdot U = 4\pi\rho \quad (16)$$

$$\rho(t, x) = \int dv f(t, x, v),$$

In Eqs. (16) t is the time and x and v are the position and the velocity of the stars respectively. $U = U(t, x)$ is the average Newtonian potential generated by the stars. The system (16) represents the non-relativistic kinetic model for an ensemble of particles (stars in the galaxy) with no collisions. The stars interact only through the gravitational forces which they generate collectively, are considered as pointlike particles, and we neglect the relativistic effects [30, 32-34]. The function $f(t, x, v)$ in the Vlasov-Poisson system (16) is non-negative and gives the density on phase space of the stars within the galaxy [30].

In this approach, the first order solutions of the Klein-Gordon equation (3) for the Ricci curvature scalar are considered like galactic high energy *scalarons*, which are expressed in terms of wave-packets having stationary solutions within the Vlasov system [30]. The energy of the wave-packet is interpreted like the Dark Matter component which guarantees the galaxy's equilibrium [30]. This approximation is not as precise as one would expect [30], but here we consider it as the starting point of our analysis.

The analysis in [30] permits to rewrite the Vlasov-Poisson system in spherical coordinates as

$$-\frac{d^2 b\tilde{R}}{dt^2} + \frac{1}{r^2} \frac{d}{dr} \left(\frac{d}{dr} b\tilde{R} r^2 \right) = (1 + 2b\tilde{R})\mu(t, r), \quad (17)$$

$$\mu(t, r) = \int \frac{dp}{\sqrt{1+p^2}} f(t, x, p), \quad (18)$$

$$\partial_t f + \frac{p}{\sqrt{1+p^2}} \cdot \partial_x f - \left[\left(\frac{d}{dt} b\tilde{R} + \frac{x \cdot p}{\sqrt{1+p^2}} \frac{1}{r} \frac{d}{dr} b\tilde{R} \right) p + \frac{x}{\sqrt{1+p^2}} \frac{1}{r} \frac{d}{dr} b\tilde{R} \right] \cdot \partial_p f = 0, \quad (19)$$

In Eqs. (17), (18) and (19) p denotes the vector $p = (p_1, p_2, p_3)$ with $p^2 = |p|^2$, and x denotes the vector $x_i = (x_1, x_2, x_3)$ [30].

As one is interested in stationary states, one calls λ the wavelength of the “galactic” gravitational wave (7), i.e. the characteristic length of the gravitational perturbation [30]. One further assumes that $\lambda \gg d$, d being the galactic scale of order $d \sim 10^5$ light-years [35, 36]. In other words, the gravitational perturbation can be considered “frozen-in” with respect to the galactic scale [30].

In that way, one can write down the system of equations defining the stationary solutions of eqs. (17), (18) and (19) [30]

$$\frac{1}{r^2} \frac{d}{dr} \left(\frac{d}{dr} b\tilde{R} r^2 \right) = (1 + 2b\tilde{R})\mu(r), \quad (20)$$

$$\mu(r) = \int \frac{dp}{\sqrt{1+p^2}} f(x, p), \quad (21)$$

$$p \cdot \partial_x f - \frac{1}{r} \frac{d}{dr} b \tilde{R} [(p \cdot x)p + x] \cdot \partial_p f = 0, \quad (22)$$

Therefore, the idea in [30] is that the spin-zero degree of freedom arising from the R^2 term in the gravitational Lagrangian, i.e. the scalaron, is a potential candidate for the dark matter. In this approach, the dominant contribution to the curvature within a galaxy comes from the scalaron field equation (3). That equation has a proper baryon source term. This enables the baryons themselves to evolve obeying a collisionless Boltzmann equation. Then the baryons can propagate on the curvature generated by the scalaron.

3 The Jeans theorem and the “Tolman-Oppenheimer-Volkov equation”

The following results will be obtained adapting the ideas introduced in [32-34,37]. Let us start by recalling some important definitions in the conformal frame:

$$\begin{aligned} P(r) &\equiv \int \frac{dp}{\sqrt{1+p^2}} \left(\frac{x \cdot p}{r} \right)^2 f(x, p) && \text{radial pressure} \\ \rho(r) &\equiv (1 + 2bR) \int dp \sqrt{1+p^2} f(x, p) && \text{mass - energy density} \\ P_T(r) &\equiv \int \frac{dp}{\sqrt{1+p^2}} \left| \frac{x \wedge p}{r} \right| f(x, p) && \text{tangential pressure.} \end{aligned} \quad (23)$$

Thus, Eq. (20) becomes

$$\frac{1}{r^2} \frac{d}{dr} \left(\frac{d}{dr} b \tilde{R} r^2 \right) = \rho(r) - P(r) - 2P_T(r). \quad (24)$$

Let us consider the stationary solution of our stellar dynamics model, i.e. Eqs. (20), (21) and (22). The particle density is a functional of the invariants of the Vlasov equation (14). The Jeans theorem states that any of these invariants must be a functional of the local energy and the angular momentum. In that way, the particle density depends only on these two integrals of the system under consideration. In order to prove these statements, one introduces the new coordinates

$$\begin{aligned} r &= |x| \\ Y &= \left(1 + \frac{b}{2} \tilde{R} \right) \frac{x \cdot p}{|x|} \\ Z &= \left(1 + b \tilde{R} \right) \left[|x|^2 - |p|^2 - (x \cdot p)^2 \right]. \end{aligned} \quad (25)$$

We note that, f being spherically symmetric, one can write it as a function of r, Y, Z , i.e. $f(x, p) \rightarrow f(r, Y, Z)$. Thus, Eq. (22) in the new coordinates reads

$$Y \partial_r f + \left[\frac{Z}{r^3} - \left(1 + b\tilde{R}\right) \frac{d}{dr} b\tilde{R} \right] \partial_Y f = 0. \quad (26)$$

This equation has the same form of eq. (2.11) in [37] with $m(r) = \left(1 + b\tilde{R}\right) \left(\frac{d}{dr} b\tilde{R} r^2\right)$. Thus, by applying the result in [37], one obtains that f must have the form

$$f(r, Y, Z) = A(\bar{E}, Z), \quad (27)$$

where

$$\bar{E}(r, Y, Z) = \frac{1}{2}Y^2 + \frac{1}{2}\frac{Z}{r^2} + \frac{1 + b\tilde{R}}{2} = \left(\frac{1 + b\tilde{R}}{2}\right) (1 + p^2). \quad (28)$$

Returning to the system of Eqs. (20), (21) and (22), one gets immediately

$$f(x, p) = A(E, X), \quad (29)$$

where

$$E = \left(1 + \frac{b}{2}\tilde{R}\right) \sqrt{1 + p^2} \quad (30)$$

is the local energy of the particles and

$$X = \left(1 + \frac{b}{2}\tilde{R}\right) |x \wedge p|^2 \quad (31)$$

is the modulus squared of the local angular momentum. Using Eq. (29) and a transformation of variables, the system (23) becomes

$$\begin{aligned} P(r) &= \frac{\pi}{r^2} \int dE \int dX A(E, X) \sqrt{E^2 - \frac{X}{r^2} - \left(1 + \frac{b}{2}\tilde{R}\right)} \\ \rho(r) &= \frac{\pi}{r^2} \int dE E^2 dX \frac{A(E, X)}{\sqrt{E^2 - \frac{X}{r^2} - \left(1 + \frac{b}{2}\tilde{R}\right)}} \\ P_T(r) &= \frac{\pi}{2r^4} \int dE \int dX \frac{X A(E, X)}{\sqrt{E^2 - \frac{X}{r^2} - \left(1 + \frac{b}{2}\tilde{R}\right)}}. \end{aligned} \quad (32)$$

A direct computation permits to obtain

$$\frac{d}{dr} P(r) = - \left(1 + b\tilde{R}\right) \frac{d}{dr} \rho(r) - \frac{2}{r} [P(r) - P_T(r)], \quad (33)$$

which is the analogous of the historical Tolman-Oppenheimer-Volkov equation [38, 39] for the system under analysis in this paper.

For the sake of completeness, we stress that the Jeans instability (gravitational stability) has been analysed in the context of modified theories of gravity

for rotating/non-rotating configurations in [40-42]. Related questions on the Tolman-Oppenheimer-Volkov equation and the Jeans instability (but not for plane waves) were also studied in [43-45]. The Tolman-Oppenheimer-Volkov equation has been also analysed in the dimensional gravity's rainbow in the presence of cosmological constant in [46] and in the framework of dilaton gravity in [47].

4 Two additional models

For the sake of completeness, we play the devil's advocate and consider two models which argue that Dark Matter is not an essential element, even though popular models postulate that it comprises roughly a fourth of a universe. Our starting point is the relation [48, 49]

$$G = G_0 \left(1 - \frac{t}{t_0} \right) \quad (34)$$

where G_0 is the present value of G , t_0 is the present age of the universe and t the time elapsed from the present epoch. Similarly one could deduce that [49]

$$r = r_0 \left(\frac{t_0}{t_0 + t} \right). \quad (35)$$

In this scheme the gravitational constant G varies slowly with time. This is suggested by Sidharth's 1997 cosmology [48], which correctly predicted a Dark Energy driven accelerating universe at a time when the accepted paradigm was the Standard Big Bang cosmology in which the universe would decelerate under the influence of dark matter.

We reiterate the following: The problem of galactic rotational curves [49, 50]. We would expect, on the basis of straightforward dynamics that the rotational velocities at the edges of galaxies would fall off according to

$$v^2 \approx \frac{GM}{r}. \quad (36)$$

However it is found that the velocities tend to a constant value,

$$v \sim 300 \text{ km/sec}. \quad (37)$$

This as known had lead to the postulation of as yet undetected additional matter, the so called Dark Matter. We observe that from Eq. (35) it can be easily deduced that [51]

$$a \equiv (\ddot{r}_0 - \ddot{r}) \approx \frac{1}{t_0} (t\ddot{r}_0 + 2\dot{r}_0) \approx -2\frac{r_0}{t_0^2}, \quad (38)$$

as we are considering infinitesimal intervals t and nearly circular orbits. Equation (38) shows that there is an anomalous inward acceleration, as if there is an extra attractive force, or an additional central mass [52].

Thus,

$$\frac{GMm}{r^2} + \frac{2mr}{t_0^2} \approx \frac{mv^2}{r} \quad (39)$$

From Eq. (39) it follows that

$$v \approx \left(\frac{2r^2}{t_0^2} + \frac{GM}{r} \right)^{1/2} \quad (40)$$

Eq. (40) shows that at distances within the edge of a typical galaxy, that is $r < 10^{23} \text{ cm}$, the equation (36) holds but as we reach the edge and beyond, that is for $r \geq 10^{24} \text{ cm}$, we have $v \sim 10^7 \frac{\text{cm}}{\text{sec}}$, in agreement with eq. (37).

Then, the time variation of G explains observation without invoking dark matter. It may also be mentioned that other effects like the Pioneer anomaly and shortening of the period of binary pulsars can be deduced [53], while new effects also are predicted.

Milgrom [54] approached the problem by modifying Newtonian dynamics at large distances. This approach is purely phenomenological. The idea was that perhaps standard Newtonian dynamics works at the scale of the solar system but at galactic scales involving much larger distances, the situation might be different. However a simple modification of the distance dependence in the gravitation law, as pointed by Milgrom would not do, even if it produced the asymptotically flat rotation curves of galaxies. Such a law would predict the wrong form of the mass velocity relation. So Milgrom suggested the following modification to Newtonian dynamics: A test particle at a distance r from a large mass M is subject to the acceleration a given by

$$a^2/a_0 = MGr^{-2}, \quad (41)$$

where a_0 is an acceleration such that standard Newtonian dynamics is a good approximation only for accelerations much larger than a_0 . The above equation however would be true when a is much less than a_0 . Both statements can be combined in the heuristic relation

$$\mu(a/a_0)a = MGr^{-2}. \quad (42)$$

In Eq. (42); $\mu(x) \approx 1$ when $x \gg 1$, and, $\mu(x) \approx x$ when $x \ll 1$. It is worthwhile to note that (41) or (42) are not deduced from any theory, but rather are an ad hoc fit to explain observations. Interestingly it must be mentioned that most of the implications of Modified Newton Dynamics (MOND) do not depend strongly on the exact form of μ .

It can then be shown that the problem of galactic velocities is solved [55-59].

It is interesting to note that there is an interesting relationship between the varying G approach, which has a theoretical base and the purely phenomenological MOND approach. Let us write

$$\beta \frac{GM}{r} = \frac{r^2}{t_0^2} \text{ or } \beta = \frac{r^3}{GMt_0^2}. \quad (43)$$

Hence,

$$\alpha_0 = v^2/r = \frac{GM}{r^2} \alpha = \frac{r}{t_0^2}. \quad (44)$$

So that

$$\frac{\alpha}{\alpha_0} = \frac{r^3}{GMt_0^2} = \beta \quad (45)$$

At this stage we can see a similarity with MOND. For if $\beta \ll 1$ we are with the usual Newtonian dynamics and if $\beta > 1$ then we get back to the varying G case exactly as with MOND.

5 Concluding remarks

The results in [30] have shown that spherically symmetric stationary states can be used as a model for galaxies in the framework of the linearized R^2 gravity. Those states could, in principle, be a partial solution to the Dark Matter Problem. In this paper, an improvement of this work has been discussed. As the star density is a functional of the invariants of the associated Vlasov equation, it has been shown that any of these invariants is in turn a functional of the local energy and the angular momentum. Then, the star density depends only on these two integrals of the Vlasov system. This result represents the so called “Jeans theorem”. In addition, an analogous of the historical Tolman-Oppenheimer-Volkov equation [38, 39] for the system considered in this paper has been discussed. We tried this extension of previous work in [30] because, on one hand, the Jeans theorem is important in galaxy dynamics and in the framework of molecular clouds [60]. On the other hand, the historical Tolman-Oppenheimer-Volkov equation constrains the structure of a spherically symmetric body of isotropic material which is in static gravitational equilibrium, as modelled by metric theories of gravity, starting from the general theory of relativity [38, 39]. Thus, a viable extended theory of gravity, like the R^2 gravity, must show consistence with these two important issues.

For the sake of completeness, in Section 4 of this paper, two additional models which argue that Dark Matter could not be an essential element have been discussed. In fact, Dark Matter is considered a mysterious and controversial issue. There is indeed a minority of researchers who think that the dynamics of galaxies could not be determined by massive, invisible dark matter halos, see [10, 11] and [51-59]. We think that, at the present time, there is not a final answer to the Dark Matter issue. In other words, it is undoubtedly true that the Universe exhibits a plethora of mysterious phenomena for which many unanswered questions still exist. Dark Matter is an important part of this intriguing puzzle. Thus, when one works in classical, modern, and developing astrophysical and cosmological theories, it is imperative to repeatedly question their capabilities, identify possible shortcomings, and propose corrections and alternative theories for experimental submission. In the procedures and practice of scientific professionals, no such clues, evidence, or data may be overlooked.

Finally, we take the chance to stress that important future impacts which could help to a better understanding of the important Dark Matter issue could arise from the nascent gravitational wave astronomy [61]. The first direct detection of gravitational waves by the LIGO Collaboration, the so called event GW150914 [61], represented a cornerstone for science and for astrophysics in particular. We hope that the gravitational wave astronomy will become an important branch of observational astronomy which will aim to use gravitational waves to collect observational data not only about astrophysical objects such as neutron stars, black holes and so on, but also about the mysterious issues of Dark Matter and Dark Energy. In order to achieve this prestigious goal, a network including interferometers with different orientations is required and we're hoping that future advancements in ground-based projects and space-based projects will have a sufficiently high sensitivity [61-63].

For the benefits of the reader, we also signal two important works on self-gravitating systems [64] and on Jeans mass for anisotropic matter [65].

6 Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

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